PARAMETRIC ANALYSIS OF THE ACCURACY OF SOLUTION OF A NONLINEAR INVERSE PROBLEM OF RECOVERING THE THERMAL CONDUCTIVITY OF A COMPOSITE MATERIAL

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The authors consider the construction of an iterative numerical algorithm and analyze the influence of the location of the temperature sensor on the accuracy of solving the inverse problem.

The wide practical use of various high-temperature structural and thermal insulation composite materials requires development of new methods of determining their thermophysical characteristics. Intense heating in these materials causes complex multistage processes which can considerably change their structure, chemical composition, and physical properties, and have a substantial influence on the characteristics of the internal heat and mass transfer. The fact that these processes are interconnected in most cases precludes the use of traditional approaches to thermophysical measurements. One meets the problem of determining the characteristics of heat and mass transfer inside composite materials under conditions close to natural [1].

One promising approach to the solution of this problem is to conduct numerical experimental thermophysical investigations, based on the processing of experimental information by the methods of inverse coefficient problems [2]. In this case it is postulated that the structure of the heat- and mass-transfer model is given, but that some of the model coefficients are known with low accuracy or are altogether unknown. One requires to use the results of measurements of thermal boundary conditions and the temperature at internal points of the test body to determine the unknown coefficients of the mathematical model.

To use the methods of coefficients of inverse heat- and mass-transfer problems in practical thermophysical investigations, one must not only create efficient numerical algorithms to solve such problems, but also develop methods of planning thermophysical experiments to be optimal in the conditions of their execution from the viewpoint of reliability of the results obtained. In particular, each specific experiment has the problem of planning the measurements, i.e., of determining the arrangement of temperature sensors in the specimen to achieve maximum accuracy in finding the specific coefficient of the inverse problem.

The present paper considers the question of construction and analysis of the efficiency of a numerical algorithm for solving an inverse problem to recover the temperature dependence of the thermal conductivity of a composite material under conditions where it undergoes unsteady heating and breakdown. A parametric mathematical model is used to investigate the accuracy indices of this algorithm as a function of the location of the temperature sensor. This approach and the results of modeling can be used to solve the problem of planning the thermophysical experiments.

We now consider a nonlinear inverse coefficient problem for the system of equations describing unsteady heating and breakdown of a composite material [1]. When we have internal temperature measurements it is convenient to represent the original system as a boundary problem in heating of a multilayer plate with identical thermophysical characteristics of the layers and zero contact thermal resistances at the boundaries between them.

In this case one must determine the temperature field $T(x, \tau)$ and the dependence of the thermal conductivity $\lambda(T)$ from the conditions

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$$c(T)\frac{\partial T_i}{\partial\tau}=\frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T_i}{\partial x}\right]-k(x,\ T)\frac{\partial T_i}{\partial x}-Q(x,\ T), \tag{1}$$

$$x_i<x<x_{i+1},\ 0<\tau\leqslant\tau_m,\ i=\overline{1,\ N-1},\ x_1=0,\ x_N=b, \tag{2}$$

$$T_1(0,\ \tau)=f_1(\tau),\ 0<\tau\leqslant\tau_m, \tag{3}$$

$$\left.\begin{aligned}&T_{i-1}(x_i,\ \tau)=T_i(x_i,\ \tau),\\&\frac{\partial T_{i-1}(x_i,\ \tau)}{\partial x}=\frac{\partial T_i(x_i,\ \tau)}{\partial x},\end{aligned}\right\}\ i=\overline{2,\ N-1}, \tag{4}$$

$$T_{N-1}(b,\ \tau)=f_N(\tau),\ 0<\tau\leqslant\tau_m, \tag{5}$$

$$T_i(x,\ 0)=\varphi_i(x),\ i=\overline{1,\ N-1}, \tag{6}$$

$$m_g(x,\ T)=\begin{cases}\int\limits_{x_r}^{x}(1-k_T)\rho_0Az^n\exp\left(-\frac{E}{RT_i}\right)dx, & T_i\geqslant T_r,\\ 0, & T_i<T_r,\end{cases} \tag{7}$$

$$T_i(x_i,\ \tau)=f_i(\tau),\ 0\leqslant\tau\leqslant\tau_m,\ i=\overline{2,\ N-1}, \tag{8}$$

where

$$k(x,\ T)=m_g(x,\ T)\frac{\partial h_g(T)}{\partial T};\ Q(x,\ T)=\frac{\partial m_g(x,\ T)}{\partial x}h_g(T);$$

$c(T)$, $h_g(T)$, $\varphi(x)$ and $f_i(\tau)$, $i=\overline{1,\ N}$ are known functions. We also consider as given the parameters describing the thermal breakdown processes in the composite material.

In solving inverse heat-conduction coefficient problems, a very efficient approach is one based on considering the problem in an extremal formulation using parameterization of the desired functions and applying the iterative methods of successive approximations [3-6]. Here as the target function we use the rms deviation of the theoretical temperatures (according to the given mathematical model) at the sensor location points from the measured values:

$$I(x_i,\ \tau,\ \lambda(T))=\sum_{i=2}^{N-1}\int\limits_0^{\tau_m}[T(x_i,\ \tau,\ \lambda(T))-f_i(\tau)]^2d\tau. \tag{9}$$

Thus, we require to determine the functions $\lambda(T)$ and $T(x,\ \tau)$ from the condition that the functional of Eq. (9) have a minimum with the constraints of Eqs. (1)-(8). This approach was considered in [6], as applied to a composite material.

As a result of parameterization of the desired function $\lambda(T)$, the original nonlinear variational problem reduces to the problem of finding the vector of the parameters from the condition of a minimum of the quality criterion of Eq. (9). For the parametrization, it is convenient to represent the functions in terms of cubic B-splines [7]. We shall approximate the function $\lambda(T)$ in the form

$$\lambda(T)=\sum_{j=-1}^{M+1}\lambda_jB_j(T), \tag{10}$$

where λ_j are the desired parameters; $B_j(T)$ are cubic B-splines, constructed on the mesh $\omega=\{T_{\min}+kH,\ \ k=\overline{-3,\ \ M+3};\ \ H=(T_{\max}-T_{\min})/M\}$; $[T_{\min},\ T_{\max}]$ is the region of definition of the function $\lambda(T)$; and M is the number of sections of the spline approximation.

In constructing the numerical iteration algorithms for solving inverse problems it is convenient to use methods of minimizing the functional (9) characterized by an efficient start of the iterative process from some "deep" initial approximation, and by a reduced rate of convergence when approaching the minimum [8]. These requirements are met by gradient search methods, in particular the method of conjugate gradients [9]. In this case the approximations are calculated from the formula

$$\lambda^{(s+1)} = \lambda^{(s)} + \alpha_s G^{(s)}, \tag{11}$$

where

$$\lambda = \{\lambda_j\}_0^M;\ G^{(s)} = -I_\lambda^{\prime(s)} + \beta_s G^{(s-1)},\ G = \{g_j\}_0^M,\ s \geqslant 2;$$

$$\beta_1 = 0,\ \beta_s = \frac{\sum_{i=1}^{N-1} \int_0^{\tau_m} I_\lambda^{\prime(s)} (I_\lambda^{\prime(s)} - I_\lambda^{\prime(s-1)})\, d\tau}{\sum_{i=1}^{N-1} \int_0^{\tau_m} (I_\lambda^{\prime(s-1)})^2 d\tau};$$

α_s is the depth of the search; and s is the iteration number.

The gradient of the functional is calculated using the solution of the boundary problem conjugate to the original problem of Eqs. (1)-(8), on the basis of analysis of the necessary conditions for the functional of Eq. (9) to be stationary [10]. In this case the conjugate boundary problem has the form

$$-c \frac{\partial \psi_i}{\partial \tau} = \lambda \frac{\partial^2 \psi_i}{\partial x^2} + k \frac{\partial \psi_i}{\partial x} - \frac{\partial Q}{\partial T} \psi_i,\quad x_i < x < x_{i+1},$$

$$0 < \tau \leqslant \tau_m,\ i = \overline{1, N-1},\ x_1 = 0,\ x_N = b, \tag{12}$$

$$\psi_1(0, \tau) = 0,\ 0 \leqslant \tau \leqslant \tau_m, \tag{13}$$

$$\left.\begin{aligned} &\psi_{i-1}(x_i, \tau) = \psi_i(x_i, \tau), &(14)\\ &\frac{\partial \psi_{i-1}(x_i, \tau)}{\partial x} - \frac{\partial \psi_i(x_i, \tau)}{\partial x} = \frac{2}{\lambda}[T_i(x_i, \tau) - f_i(\tau)] &(15) \end{aligned}\right\} i = \overline{2, N-1},$$

$$\psi_{N-1}(b, \tau) = 0,\ 0 \leqslant \tau \leqslant \tau_m, \tag{16}$$

$$\psi_i(x, \tau_m) = 0,\ i = \overline{1, N-1}. \tag{17}$$

Taking account of approximation (10) and using the solution of the problem of Eqs. (12)-(17), we can obtain the following relations for the components of the vector of the gradient of the target functional:

$$I'_{\lambda_j} = \frac{\partial I}{\partial \lambda_j} = \sum_{i=1}^{N-1} \int_0^{\tau_m} \int_{x_i}^{x_{i+1}} \psi_i \left[\frac{dB_j(T)}{dT_i} \left(\frac{\partial T_i}{\partial x} \right)^2 + B_j(T) \frac{\partial^2 T_i}{\partial x^2} \right] dx d\tau,\ j = \overline{1, M+1}. \tag{18}$$

It is efficient, from the viewpoint of reducing expenditure of machine time, to use a linear estimate of the search pitch α. In this case the value of α can be determined from the formula (see [6])

$$\alpha_s = -\frac{\sum_{i=2}^{N-1} \int_0^{\tau_m} [T_i(x_i, \tau, \lambda^{(s)}) - f_i(\tau)]\, \vartheta_i(x_i, \tau, G^{(s)})\, d\tau}{\sum_{i=2}^{N-1} \int_0^{\tau_m} [\vartheta_i(x_i, \tau, G^{(s)})]^2 d\tau}, \tag{19}$$

where $\vartheta_i(x_i, \tau, G^{(s)})$ is the solution of the boundary problem for the temperature increment:

$$c\frac{\partial\vartheta_i}{\partial\tau} = \lambda\frac{\partial^2\vartheta_i}{\partial x^2} + \left(2\frac{\partial\lambda}{\partial x} - k\right)\frac{\partial\vartheta_i}{\partial x} + \left(\frac{\partial^2\lambda}{\partial x^2} - \frac{\partial k}{\partial x} - \frac{\partial Q}{\partial T} - \frac{\partial c}{\partial\tau}\right)\vartheta_i + \frac{\partial}{\partial x}\left(\Delta\lambda\frac{\partial T_i}{\partial x}\right),\ x_i < x < x_{i+1},$$

$$0 < \tau \leqslant \tau_m,\ i = \overline{1,\ N-1},\ x_1 = 0,\ x_N = b, \tag{20}$$

$$\vartheta_1(0,\ \tau) = 0,\ 0 \leqslant \tau \leqslant \tau_m, \tag{21}$$

$$\left.\begin{aligned} &\vartheta_{i-1}(x_i,\ \tau) = \vartheta_i(x_i,\ \tau), \qquad (22)\\ &\frac{\partial\vartheta_{i-1}(x_i,\ \tau)}{\partial x} = \frac{\partial\vartheta_i(x_i,\ \tau)}{\partial x}, \qquad (23)\end{aligned}\right\}\ i = \overline{2,\ N-1},$$

$$\vartheta_{N-1}(b,\ \tau) = 0,\ 0 \leqslant \tau \leqslant \tau_m, \tag{24}$$

$$\vartheta_i(x,\ 0),\ i = \overline{1,\ N-1}, \tag{25}$$

$$\Delta\lambda = \sum_{j=1}^{M+1}\frac{\partial\lambda(T)}{\partial\lambda_j}\Delta\lambda_j.$$

It is expedient to use the numerical methods of [11] to solve the boundary problems of (1)-(8), (12)-(17) and (20)-(25).

We shall conduct a further analysis of the accuracy of the algorithm described under conditions where the minimum experimental information is given, assuming that the temperature measurements are made at the boundaries and at one internal point of the test specimen. It was shown in [12] that the problem of recovering one coefficient in the homogeneous nonuniform heat-conduction equation from this information has a unique solution. The practical investigation made in [6] has confirmed the hypothesis that there is a unique solution to the more complex inverse problem (1)-(9). In solving an incorrectly posed problem it is important to stop the iteration process. In the case where exact values of the "input" temperatures are known, to cut short the iterative process of searching for $\lambda(T)$ the condition

$$\max_{\tau}\,[\mathrm{abs}\,(T_2(x_2,\ \tau,\ \lambda^{(s)}) - f_2(\tau))] \leqslant \varepsilon_T$$

is used, where $\varepsilon_T > 0$ is the error in calculating the temperature profile at the points where thermocouples are installed.

In the case when the "input" temperatures are given with an error, the time for stopping the iteration process is determined by the condition that the functional of Eq. (9) that is minimized be equal to the integral error of the temperature measurement

$$I(x_2,\ \tau,\ \lambda^{(s)}) \leqslant \delta^2 \leqslant \int_0^{\tau_m}[T_2(x_2,\ \tau,\ \lambda^{(s)}) - f_2(\tau)]^2 d\tau, \tag{26}$$

where $\delta^2 = \int_0^{\tau_m}\sigma^2(\tau)\,d\tau$ is the estimate of the generalized error of the input data; $\sigma^2(\tau)$ is the rms deviation of the temperatures at the point x_2 at time τ.

Here it is assumed that the parameters of the different mesh in the numerical solution of the problem of Eqs. (1)-(8) are chosen so that the errors can be neglected in comparison with the quantity δ^2. Round-off errors are also not taken into account in the computer calculation process.

It was shown in [4, 13] that iteration algorithms like that described above with the stop condition of Eq. (26) in the linear case are regularizing, and can be used to solve incorrect problems. The experience of using iterative algorithms to solve for the coefficients of inverse problems [3-6] has shown them to have high efficiency even in nonlinear cases.

The modeling was done as follows. The exact data on the "measured" temperatures of Eq. (8) at various distances from the heated surface were obtained by solving the direct problem of Eqs. (1)-(7) with given thermal boundary conditions of the second kind. The values of all the coefficients and parameters describing the heat- and mass-transfer process in the model were considered known, and corresponded to a typical semiorganic polymer material. The computed dependences of temperature on time at an internal point of the specimen with given coordinate x_2 and at its boundaries were then used as input data in solving the problem to determine the "unknown" dependence $\lambda(T)$. The initial approximation $\lambda^{(1)}(T)$ was assigned arbitrarily.

To solve the inverse problem in the perturbed input data the errors in the experimental function $f_2(\tau)$ were modeled according to a uniform distribution law of probability density of perturbations with an error not exceeding 3% of the maximum value of the temperature. With this kind of approach, one can model an experiment in which the temperature is measured with the help of thermocouples and one immediately compares the known dependence $\lambda^*(T)$ with the value recovered from solving the inverse problem in the various computational cases.

As the exact value of the recovered dependence of the thermal conductivity on the temperature we considered the polynomial

$$\lambda^*(T) = 6{,}7\cdot 10^{-7}T^2 - 6{,}08\cdot 10^{-4}T + 0{,}217 \left[\frac{\mathrm{W}}{\mathrm{m\ deg\ K}}\right].$$

The average relative error in the recovered function $\lambda(T)$ from solving the inverse problem was estimated from the formula

$$\bar{\varepsilon}_\lambda = \frac{1}{l}\sum_{k=1}^{l} \mathrm{abs}\,[(\lambda(T_k) - \lambda^*(T_k))/\lambda^*(T_k)], \tag{27}$$

where λ^*, λ are the exact value of the desired function and the value recovered from solving the inverse problem, respectively; l is the number of nodes in the interval $[T_{min}, T_{max}]$ at which one compares the values λ and λ^*. The number of sections of the spline approximation to the desired function was taken as three.

Figure 1 shows the dependence $\lambda(T)$ recovered using the above algorithm to solve the inverse problem, for exact and perturbed values of the input temperature. For the case of perturbed input data Fig. 2 shows values of the temperatures $T(x_2, \tau)$ recovered from solution of the inverse problem. We note that in solving the inverse problem with exact input data the difference in the recovered values of temperature did not exceed 0.3°K. The dimensionless

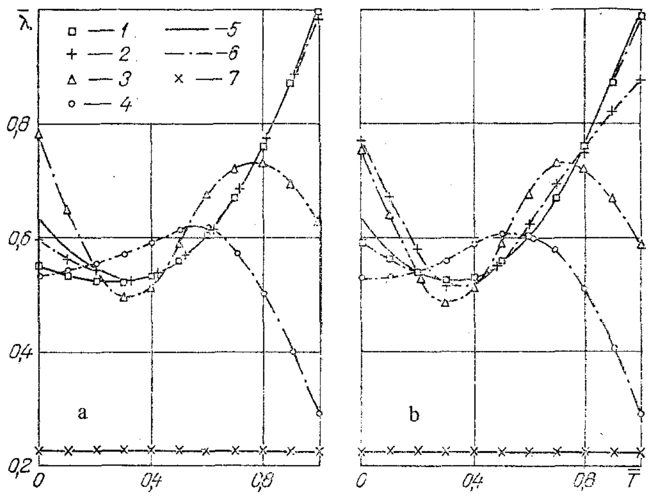

Fig. 1. Recovery of the dependence $\lambda(T)$ at the exact values (a) and the perturbed values (b) of the input data for various values of the coordinates of thermocouple location ($x_2\cdot 10^3$ m): 1) x_2 = 0.7; 2) 1.4; 3) 2.1; 4) 4.2; 5) exact values; 6) values recovered from solving the inverse problem; 7) initial approximation; λ_{max} = 0.149 W/(m·K), T_0 = 303 K, T_{max} = 780 K.

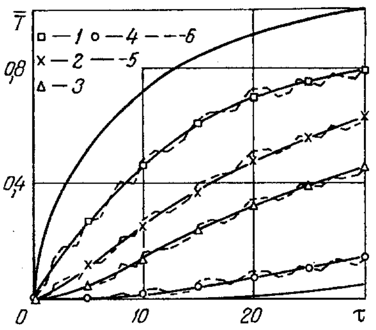

Fig. 2

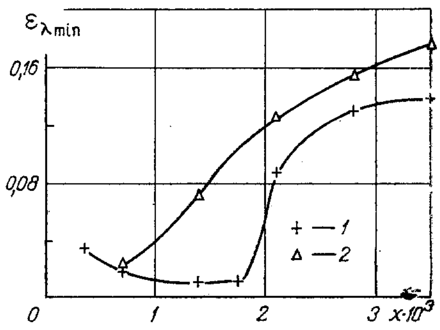

Fig. 3

Fig. 2. Temperatures at the thermocouple location points ($x \cdot 10^3$ m): 1) $x_2 = 0.7$; 2) 1.4; 3) 2.1; 4) 4.2; 5) exact values; 6) perturbed values; T_0 = 303 K, T_{max} = 780 K.

Fig. 3. The minimum relative error in recovering the thermal conductivity: 1) exact; 2) perturbed input data. x is in m.

coordinates in Figs. 1 and 2 are $\bar{T}=(\dot{T}-T_0)/(T_{max}-T_0), \bar{\lambda}=\lambda/\lambda_{max}$.

From the results of the mathematical modeling one can analyze the sensitivity of the algorithm for solving the inverse problem for a variation in the coordinates of thermocouple location (Fig. 3). It should be noted that analogous results were obtained also for the uniform heat-conduction equation.

The analysis showed that for successful use of the methods of inverse problems to determine the thermophysical characteristics of actual materials one must first analyze questions associated both with formulating the problem and setting up the solution algorithms, and with the choice of conditions of the experiment, to ensure reliability of the results obtained and efficient use of the proposed algorithms.

NOTATION

c, volume heat capacity; λ, thermal conductivity; m_g, specific mass flow rate; h_g, enthalpy of the gas phase of thermal breakdown products; T, temperature; x, three-dimensional coordinate; τ, time; τ_m, right-hand boundary value of the time interval; $f_i(\tau)$, input temperatures; z, concentration of the decomposable component; A, preexponential factor; n, order of the breakdown reaction; E/R, activation energy; k_T, limiting value of the coke number; ρ_0, density of the original material; T_r, temperature at the start of thermal breakdown; b, right-hand boundary value of the three-dimensional interval; I, functional; ϑ, temperature increment; ψ, conjugate variable; i, three-dimensional subscript; T_{min}, minimum value of temperature; T_{max}, maximum value of temperature; λ_{max}, maximum value of thermal conductivity.

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RECONSTRUCTING THE EFFECTIVE COEFFICIENT OF THERMAL CONDUCTIVITY OF ASBESTOS—TEXTOLITE FROM THE SOLUTION OF THE INVERSE PROBLEM

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The article examines the practical application of the algorithm for solving inverse problems in processing experimental data.

The intense development of the theory and the increasing range of application of the methods of solving inverse problems of heat exchange [1] led to their widespread use in thermophysical research [2-5]. Such an approach in the investigation of the thermophysical characteristics of high-temperature composite materials under nonsteady conditions solves the problem of modeling the structure of the material and the nature of how internal processes proceed [6], and moreover, it makes it possible to determine these characteristics for mathematical models in which their application is assumed.

Sometimes the problem of determining the effective values of thermophysical characteristics may be examined; the use of these characteristics makes it possible to generalize in fairly simple form the results of experimental investigations. Furthermore, such characteristics may be used for calculating temperature fields of coatings in the range of change of external conditions that is of interest to the researcher.

The principal object of the present work consists in investigating the possibility of the practical application of the methods of inverse problems for determining the thermophysical characteristics of composite materials under nonsteady conditions.

We analyze the errors connected with thermocouple temperature measurements in high-temperature decomposing material, and the accuracy of the obtained results is evaluated. For processing the experimental data we used the algorithm for solving inverse coefficient problems of heat conduction explained in [2].

We analyzed a model of an unbounded flat plate in which at four points thermocouple measurements were carried out.

The temperature measurements at the outer points of the examined region were used as thermal boundary conditions. The input data for solving the inverse problem were the temperature measurements at the inner points of the region.

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